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## SUPERCONGRUENCES INVOLVING DUAL SEQUENCES

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ABSTRACT. In this paper we study some sophisticated supercongruences involving dual sequences. For  $n = 0, 1, 2, \dots$  define

$$d_n(x) = \sum_{k=0}^n \binom{n}{k} \binom{x}{k} 2^k$$

and

$$s_n(x) = \sum_{k=0}^n \binom{n}{k} \binom{x}{k} \binom{x+k}{k} = \sum_{k=0}^n \binom{n}{k} (-1)^k \binom{x}{k} \binom{-1-x}{k}.$$

For any odd prime  $p$  and  $p$ -adic integer  $x$ , we determine  $\sum_{k=0}^{p-1} (\pm 1)^k d_k(x)^2$  and  $\sum_{k=0}^{p-1} (2k+1) d_k(x)^2$  modulo  $p^2$ ; for example, we establish the new supercongruence

$$\sum_{k=0}^{p-1} (-1)^k d_k(x)^2 \equiv (-1)^{\langle x \rangle_p} \pmod{p^2},$$

where  $\langle x \rangle_p$  denotes the least nonnegative integer  $r$  with  $x \equiv r \pmod{p}$ . For any prime  $p > 3$  and  $p$ -adic integer  $x$ , we determine  $\sum_{k=0}^{p-1} s_k(x)^2$  modulo  $p^2$  (or  $p^3$  if  $x \in \{0, \dots, p-1\}$ ), and show that

$$\sum_{k=0}^{p-1} (2k+1) s_k(x)^2 \equiv 0 \pmod{p^2}.$$

We also pose several related conjectures; for example, we conjecture that for any positive integer  $n$  the polynomials

$$\frac{x(x+1)}{2n^2} \sum_{k=0}^{n-1} (2k+1) d_k(x)^2 \quad \text{and} \quad \frac{1}{n^2} \sum_{k=0}^{n-1} (2k+1) s_k(x)^2$$

are both integer-valued.

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## 1. INTRODUCTION

For a sequence of numbers  $a_0, a_1, a_2, \dots$ , its *dual sequence*  $a_0^*, a_1^*, a_2^*, \dots$  is given by

$$a_n^* = \sum_{k=0}^n \binom{n}{k} (-1)^k a_k \quad (n = 0, 1, 2, \dots). \quad (1.1)$$

It is well-known that  $(a_n^*)^* = a_n$  for all  $n = 0, 1, 2, \dots$ . One may consult [Su03] for some combinatorial identities involving dual sequences. The author [Su14b, Theorem 2.2] showed that if  $p$  is an odd prime,  $a_0, a_1, \dots, a_{p-1}$  are  $p$ -adic integers and  $m$  is an integer with  $p \nmid m(m-4)$  then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} a_k^*}{(4-m)^k} \equiv \left( \frac{m(m-4)}{p} \right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k} a_k}{m^k} \pmod{p},$$

where  $\left( \frac{\cdot}{p} \right)$  denotes the Legendre symbol.

Let  $p$  be any odd prime  $p$ . The author and R. Tauraso [ST, (1.9)] showed that

$$\sum_{k=0}^{p-1} \binom{2k}{k} \equiv \left( \frac{p}{3} \right) \pmod{p^2},$$

where  $(-)$  denotes the Legendre symbol. In [Su11b] the author determined  $\sum_{k=0}^{p-1} \binom{2k}{k} / m^k$  modulo  $p^2$  for any integer  $m \not\equiv 0 \pmod{p}$ , and moreover he proved that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{2^k} \equiv \left( \frac{-1}{p} \right) - p^2 E_{p-3} \pmod{p^3},$$

where  $E_0, E_1, E_2, \dots$  are the Euler numbers given by

$$E_0 = 1 \quad \text{and} \quad \sum_{\substack{k=0 \\ 2|k}}^n \binom{n}{k} E_{n-k} = 0 \quad \text{for } n \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}.$$

If  $a_n = \binom{2n}{n} = \binom{-1/2}{n} (-4)^n$  for  $n \in \mathbb{N} = \{0, 1, 2, \dots\}$ , then we have  $a_n^* = (-1)^n T_n$  for all  $n \in \mathbb{N}$  by [G, (3.86)], where the central trinomial coefficient  $T_n$  is the constant term of  $(1+x+x^{-1})^n$ . In [Su14b] the author showed that

$$\sum_{k=0}^{p-1} T_k^2 \equiv \left( \frac{-1}{p} \right) \pmod{p}.$$

For any  $n \in \mathbb{N}$  we have the useful Chu-Vandermonde identity (cf. [G, (3.1)])

$$\sum_{k=0}^n \binom{x}{k} \binom{y}{n-k} = \binom{x+y}{n}. \quad (1.2)$$

Thus, if  $a_k = (-1)^k \binom{x}{k}$  for all  $k \in \mathbb{N}$ , then for any  $n \in \mathbb{N}$  we have

$$\begin{aligned} a_n^* &= \sum_{k=0}^n \binom{n}{k} (-1)^k \left( (-1)^k \binom{x}{k} \right) \\ &= \sum_{k=0}^n \binom{n}{n-k} \binom{x}{k} = \binom{n+x}{n} = (-1)^n \binom{-1-x}{n}. \end{aligned}$$

and hence  $a_n a_n^* = \binom{x}{n} \binom{-1-x}{n}$ . In particular, for the sequence  $a_n = (-1)^n \binom{-1/2}{n} = \binom{2n}{n}/4^n$  ( $n = 0, 1, 2, \dots$ ) we have  $a_n^* = a_n$  for all  $n \in \mathbb{N}$ .

In 2003 Rosriguez-Villegas [RV] made conjectures on  $\sum_{k=0}^{p-1} \binom{x}{k} \binom{-1-x}{k}$  modulo  $p^2$  with

$$x \in \left\{ -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6} \right\}$$

for any prime  $p > 3$ , and they were later proved by E. Mortenson [M1, M2]. In [Su11b] the author proved that

$$\sum_{k=0}^{p-1} \binom{-1/2}{k}^2 = \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k} \equiv \left( \frac{-1}{p} \right) - p^2 E_{p-3} \pmod{p^3} \quad (1.3)$$

for any prime  $p > 3$ . During their study of special values of spectral zeta functions, K. Kimoto and M. Wakayama [KW] defined

$$\tilde{J}_2(n) := \sum_{k=0}^n \binom{n}{k} (-1)^k \binom{-1/2}{k}^2 \quad \text{for } n = 0, 1, 2, \dots,$$

(cf. [KW, (3.4)]) and conjectured that

$$\sum_{n=0}^{p-1} \tilde{J}_2(n)^2 \equiv \left( \frac{-1}{p} \right) \pmod{p^3} \quad (1.4)$$

for any odd prime  $p$ . In 2014 L. Long, R. Osburn and H. Swisher [LOS] confirmed (1.4) for any odd prime  $p$ .

Motivated by the above work, we give the following definition.

*Definition 1.1.* For any  $n \in \mathbb{N}$  we define the polynomials

$$D_n(x, y) := \sum_{k=0}^n \binom{n}{k} \binom{x}{k} y^k \quad \text{and} \quad S_n(x, y) := \sum_{k=0}^n \binom{n}{k} \binom{x}{k} \binom{-1-x}{k} y^k, \quad (1.5)$$

and set

$$d_n(x) := d_n(x, 2) \quad \text{and} \quad s_n(x) := S_n(x, -1) = \sum_{k=0}^n \binom{n}{k} \binom{x}{k} \binom{x+k}{k}. \quad (1.6)$$

Let  $n \in \mathbb{N}$ . In view of the Chu-Vandermonde identity (1.2), we have

$$D_n(x, 1) = \sum_{k=0}^n \binom{n}{n-k} \binom{x}{k} = \binom{n+x}{n} = (-1)^n \binom{-1-x}{n}.$$

Thus

$$\sum_{k=0}^{p-1} D_k(x, 1) D_k(-1-x, 1) = \sum_{k=0}^{p-1} \binom{x}{k} \binom{-1-x}{k} \equiv (-1)^{\langle x \rangle_p} \pmod{p^2}$$

for any prime  $p > 3$  and  $p$ -adic integer  $x$  (cf. [S1]), where  $\langle x \rangle_p$  denotes the least  $r \in \mathbb{N}$  with  $x \equiv r \pmod{p}$ . Note that those  $d_n(m)$  with  $m, n \in \mathbb{N}$  are called Delannoy numbers in combinatorics. As discovered by Delannoy, the number  $d_n(m)$  is just the number of lattice paths from  $(0, 0)$  to  $(m, n)$  in which only east  $(1, 0)$ , north  $(0, 1)$ , and northeast  $(1, 1)$  steps are allowed (cf. R. P. Stanley [St, p. 178]).

Our main goal of this paper is to investigate supercongruences involving the polynomials  $d_n(x)$  and  $s_n(x)$ . Clearly  $s_n(x) = s_n(-1-x)$  for all  $n = 0, 1, 2, \dots$ . We mention that

$$d_n(x) = (-1)^n d_n(-1-x) \quad \text{for all } n \in \mathbb{N}, \quad (1.7)$$

which will be explained in Remark 2.1.

We first present a general theorem on congruences involving  $D_n(x, y)$  and  $S_n(x, y)$  modulo primes.

**Theorem 1.1.** *Let  $p$  be a prime and let  $x$  and  $y$  be  $p$ -adic integers with  $y \not\equiv 0 \pmod{p}$ .*

(i) *We have*

$$\sum_{k=0}^{p-1} D_k(x, y) D_k(-1-x, y) \equiv (-1)^{\langle x \rangle_p} \pmod{p}. \quad (1.8)$$

*If  $y \not\equiv 1 \pmod{p}$ , then*

$$\sum_{k=0}^{p-1} (1-y)^k D_k(x, y) D_k\left(x, \frac{y}{y-1}\right) \equiv (-1)^{\langle x \rangle_p} \pmod{p}. \quad (1.9)$$

(ii) *Provided  $p \neq 2$ , we have*

$$\sum_{k=0}^{p-1} S_k(x, y)^2 \equiv \begin{cases} \left(\frac{-1}{p}\right) \pmod{p} & \text{if } x \equiv -1/2 \pmod{p}, \\ 0 \pmod{p} & \text{otherwise.} \end{cases} \quad (1.10)$$

**Theorem 1.2.** *Let  $p$  be any odd prime and let  $x$  be a  $p$ -adic integer. Then*

$$\sum_{k=0}^{p-1} (-1)^k d_k(x)^2 = \sum_{k=0}^{p-1} d_k(x) d_k(-1-x) \equiv (-1)^{\langle x \rangle_p} \pmod{p^2}, \quad (1.11)$$

$$\begin{aligned} \sum_{k=0}^{p-1} d_k(x)^2 &= \sum_{k=0}^{p-1} (-1)^k d_k(x) d_k(-1-x) \\ &\equiv \begin{cases} \left(\frac{-1}{p}\right) \pmod{p^2} & \text{if } x \equiv -1/2 \pmod{p}, \\ (-1)^{\langle x \rangle_p} (p + 2x - 2\langle x \rangle_p) / (2x + 1) \pmod{p^2} & \text{otherwise,} \end{cases} \end{aligned} \quad (1.12)$$

and

$$\sum_{k=0}^{p-1} (2k+1) d_k(x)^2 \equiv \begin{cases} -x \pmod{p^2} & \text{if } x \equiv 0 \pmod{p}, \\ x+1 \pmod{p^2} & \text{if } x \equiv -1 \pmod{p}, \\ 0 \pmod{p^2} & \text{otherwise.} \end{cases} \quad (1.13)$$

*Remark 1.1.* (a) For any odd prime  $p$  and  $p$ -adic integer  $x$ , we observe that

$$\sum_{k=0}^{p-1} (-1)^k (2k+1) d_k(x)^2 \equiv (-1)^{\langle x \rangle_p} (2x+1) (p + 2(x - \langle x \rangle_p)) \pmod{p^2}. \quad (1.14)$$

(b) We note that

$$d_n \left( -\frac{1}{2} \right) = \sum_{k=0}^n \binom{n}{k} \frac{\binom{2k}{k}}{(-2)^k} = \begin{cases} \binom{2m}{m} / 4^m & \text{if } n = 2m \text{ for some } m \in \mathbb{N}, \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

For any prime  $p > 3$ , we actually have

$$\sum_{n=0}^{p-1} (\pm 1)^n d_n \left( -\frac{1}{2} \right)^2 = \sum_{m=0}^{(p-1)/2} \frac{\binom{2m}{m}^2}{16^m} \equiv \left( \frac{-1}{p} \right) + p^2 E_{p-3} \pmod{p^2}$$

by [Su11b, (1.7)].

**Corollary 1.1.** *Let  $p$  be an odd prime. Then*

$$\sum_{k=0}^{p-1} (-1)^k d_k \left( -\frac{1}{4} \right)^2 \equiv \left( \frac{-2}{p} \right) \pmod{p^2}, \quad \sum_{k=0}^{p-1} d_k \left( -\frac{1}{4} \right)^2 \equiv p \left( \frac{2}{p} \right) \pmod{p^2}.$$

If  $p > 3$ , then

$$\begin{aligned} \sum_{k=0}^{p-1} (-1)^k d_k \left( -\frac{1}{3} \right)^2 &\equiv \left( \frac{p}{3} \right) \pmod{p^2}, \quad \sum_{k=0}^{p-1} d_k \left( -\frac{1}{3} \right)^2 \equiv p \pmod{p^2}, \\ \sum_{k=0}^{p-1} (-1)^k d_k \left( -\frac{1}{6} \right)^2 &\equiv \left( \frac{-1}{p} \right) \pmod{p^2}, \quad \sum_{k=0}^{p-1} d_k \left( -\frac{1}{6} \right)^2 \equiv p \left( \frac{3}{p} \right) \pmod{p^2}. \end{aligned}$$

To obtain supercongruences involving  $s_n(x)$ , we need some new combinatorial identities.

**Theorem 1.3.** *Let  $m, n \in \mathbb{N}$ . Then we have the identity*

$$\begin{aligned} & (-1)^{m+n} \sum_{k=0}^n (-1)^k \binom{x}{k} \binom{y}{k+m} \binom{z}{n-k} \\ &= \sum_{k=0}^n (-1)^k \binom{x}{k} \binom{m-1-y}{k+m} \binom{n-x-z-1}{n-k}. \end{aligned} \quad (1.15)$$

Consequently, if  $x + y = -1$ , then

$$\sum_{k=0}^n (-1)^k \binom{n+d}{k+d} \binom{x}{k} \binom{y}{k} = (-1)^n \sum_{k=0}^n (-1)^k \binom{x}{k}^2 \binom{y-d}{n-k} \quad (1.16)$$

for any  $d \in \mathbb{N}$ , and we have the symmetric identity

$$\sum_{k=0}^n (-1)^k \binom{x}{k}^2 \binom{y+z}{n-k} = \sum_{k=0}^n (-1)^k \binom{y}{k}^2 \binom{x+z}{n-k}. \quad (1.17)$$

*Remark 1.2.* The author first noted (1.17) with  $z = 0$  in 2011 during his seeking for new series for  $1/\pi$  (see the Remark after Conjecture 4 of [Su11a]).

**Corollary 1.2.** *Let  $m$  be a positive integer and let  $n, x \in \{0, \dots, m-1\}$ . Then*

$$\sum_{k=0}^n \binom{n+d}{k+d} \binom{x}{k} \binom{x+k}{k} = (-1)^x \sum_{k=0}^{m-1} \binom{x}{k} \binom{-1-x}{k} \binom{n+k+d}{k+d} \quad (1.18)$$

for any  $d \in \mathbb{N}$ , in particular

$$s_n(x) = (-1)^x \sum_{k=0}^{m-1} \binom{x}{k} \binom{-1-x}{k} \binom{n+k}{k}. \quad (1.19)$$

*Remark 1.3.* For any  $k, n \in \mathbb{N}$ , we clearly have

$$\binom{n+1}{k+1} \binom{x}{k} \binom{x+k}{k} = \frac{n+1}{k+1} \binom{n}{k} \binom{x+k}{2k} \binom{2k}{k} = (n+1) \binom{n}{k} \binom{x+k}{2k} C_k,$$

where  $C_k$  denotes the  $k$ -th Catalan number  $\binom{2k}{k}/(k+1) = \binom{2k}{k} - \binom{2k}{k+1}$ .

With the help of Corollary 1.2 and some other lemmas, we are able to establish the following theorem.

**Theorem 1.4.** *Let  $p > 3$  be a prime and let  $x$  be a  $p$ -adic integer. If  $x \not\equiv -1/2 \pmod{p}$ , then*

$$\sum_{k=0}^{p-1} s_k(x)^2 \equiv (-1)^{\langle x \rangle_p} \frac{p + 2(x - \langle x \rangle_p)}{2x + 1} \pmod{p^2}. \quad (1.20)$$

Also,

$$\sum_{k=0}^{p-1} s_k(x)^2 \equiv \frac{(-1)^x p}{2x + 1} \pmod{p^3} \quad \text{for } x = 0, 1, \dots, p-1. \quad (1.21)$$

Moreover,

$$\sum_{k=0}^{p-1} (2k+1)s_k(x)^2 \equiv 0 \pmod{p^2}. \quad (1.22)$$

We will prove Theorems 1.1 and 1.2 in the next section, and show Theorem 1.3 and Corollary 1.2 in Section 3. On the basis of some lemmas presented in Section 4, we are able to prove Theorem 1.4 in Section 5. In Section 6 we pose several related conjectures for further research.

## 2. PROOFS OF THEOREMS 1.1 AND 1.2

**Lemma 2.1.** *For any  $n \in \mathbb{N}$  we have the identity*

$$D_n(x, y) = \sum_{k=0}^n \binom{n}{k} \binom{x+k}{k} y^k (1-y)^{n-k}. \quad (2.1)$$

*Proof.* This is a known result due to Ljunggren (cf. [G, (3.18)]). Nevertheless we give here a simple proof. As mentioned after (1.2), for  $k = 0, \dots, n$  we have

$$\binom{x+k}{k} = (-1)^k \binom{-1-x}{k} = \sum_{j=0}^k \binom{k}{j} (-1)^j \binom{-1-x}{j}.$$

Therefore

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} \binom{x+k}{k} y^k (1-y)^{n-k} &= \sum_{k=0}^n \binom{n}{k} \sum_{j=0}^k \binom{k}{j} \binom{x}{j} y^k (1-y)^{n-k} \\ &= \sum_{j=0}^n \binom{n}{j} \binom{x}{j} y^j \sum_{k=j}^n \binom{n-j}{k-j} y^{k-j} (1-y)^{n-k} \\ &= \sum_{j=0}^n \binom{n}{j} \binom{x}{j} y^j = D_n(x, y) \end{aligned}$$

with the help of the binomial theorem. This concludes the proof.  $\square$

*Remark 2.1.* (2.1) with  $y = 2$  yields the equality in (1.7).

*Proof of Theorem 1.1.* (i) If  $y \not\equiv 1 \pmod{p}$ , then by Lemma 2.1 we have

$$D_k(-1-x, y) = (1-y)^k D_k\left(x, \frac{-y}{1-y}\right) \quad \text{for all } k \in \mathbb{N}, \quad (2.2)$$

and hence

$$\sum_{k=0}^{p-1} (1-y)^k D_k(x, y) D_k\left(x, \frac{y}{y-1}\right) = \sum_{k=0}^{p-1} D_k(x, y) D_k(-1-x, y).$$

So, it suffices to show (1.8).

Let  $r = \langle x \rangle_p$ . For  $i, j = 0, \dots, p-1$ , we clearly have

$$\binom{x}{i} \binom{-1-x}{j} \equiv \binom{r}{i} \binom{p-1-r}{j} \pmod{p}.$$

If  $i+j > p-1$ , then  $\binom{r}{i} \binom{p-1-r}{j} = 0$  since  $i > r$  or  $j > p-1-r$ . If  $i+j = p-1$ , then  $\binom{r}{i} \binom{p-1-r}{j} = 0$  unless  $i = r$  and  $j = p-1-r$ . If  $i+j < p-1$ , then  $\binom{k}{i} \binom{k}{j}$  is a polynomial in  $k$  of degree smaller than  $p-1$  with  $p$ -adic integer coefficients, and hence

$$\sum_{k=0}^{p-1} \binom{k}{i} \binom{k}{j} \equiv 0 \pmod{p}$$

since  $\sum_{k=0}^{p-1} k^s \equiv 0 \pmod{p}$  for all  $s = 0, 1, \dots, p-2$ . Therefore, in view of Fermat's little theorem and Wilson's theorem, we have

$$\begin{aligned} & \sum_{k=0}^{p-1} D_k(x, y) D_k(-1-x, y) \\ & \equiv \sum_{k=0}^{p-1} D_k(r, y) D_k(p-1-r, y) = \sum_{k=0}^{p-1} \sum_{i=0}^k \binom{k}{i} \binom{r}{i} y^i \sum_{j=0}^k \binom{k}{j} \binom{p-1-r}{j} y^j \\ & = \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \binom{r}{i} \binom{p-1-r}{j} y^{i+j} \sum_{k=0}^{p-1} \binom{k}{i} \binom{k}{j} \\ & \equiv \binom{r}{r} \binom{p-1-r}{p-1-r} y^{r+(p-1-r)} \sum_{k=0}^{p-1} \binom{k}{r} \binom{k}{p-1-r} \\ & \equiv \sum_{k=0}^{p-1} \frac{k^{p-1}}{r!(p-1-r)!} \equiv (p-1)! \binom{p-1}{r} (p-1) \equiv (-1)^r \pmod{p}. \end{aligned}$$

This proves the desired (1.8).



(ii) Below we assume that  $p \neq 2$ . Clearly,

$$\binom{x}{j} \binom{-1-x}{j} = \binom{x}{j} \binom{x+j}{j} (-1)^j = \binom{x+j}{2j} \binom{2j}{j} (-1)^j$$

for all  $j = 0, 1, 2, \dots$ . Thus

$$\begin{aligned} \sum_{k=0}^{p-1} S_k(x, y)^2 &= \sum_{k=0}^{p-1} \sum_{i=0}^k \binom{k}{i} \binom{x}{i} \binom{-1-x}{i} y^i \sum_{j=0}^k \binom{k}{j} \binom{x}{j} \binom{-1-x}{j} y^j \\ &= \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \binom{x+i}{2i} \binom{x+j}{2j} (-y)^{i+j} \binom{2i}{i} \binom{2j}{j} \sum_{k=0}^{p-1} \binom{k}{i} \binom{k}{j}. \end{aligned}$$

Note that  $p \mid \binom{2r}{r}$  for all  $r \in \{(p+1)/2, \dots, p-1\}$ , and that  $\sum_{k=0}^{p-1} k^s \equiv 0 \pmod{p}$  for all  $s = 0, \dots, p-2$ . So, if  $i, j \in \{0, \dots, p-1\}$  and

$$\binom{2i}{i} \binom{2j}{j} \sum_{k=0}^{p-1} \binom{k}{i} \binom{k}{j} \not\equiv 0 \pmod{p},$$

then we must have  $i = j = (p-1)/2$ . Therefore,

$$\begin{aligned} \sum_{k=0}^{p-1} S_k(x, y)^2 &\equiv \binom{x + (p-1)/2}{p-1}^2 (-y)^{p-1} \binom{p-1}{(p-1)/2}^2 \frac{\sum_{k=0}^{p-1} k^{p-1}}{(p-1)!} \binom{p-1}{(p-1)/2} \\ &\equiv \left( \frac{-1}{p} \right) \binom{\langle x \rangle_p + (p-1)/2}{p-1}^2 \\ &\equiv \begin{cases} \left( \frac{-1}{p} \right) \pmod{p} & \text{if } \langle x \rangle_p = (p-1)/2, \text{ i.e., } x \equiv -1/2 \pmod{p}, \\ 0 \pmod{p} & \text{otherwise.} \end{cases} \end{aligned}$$

This proves (1.10).

By the above, we have completed the proof of Theorem 1.1.  $\square$

*Remark 2.2.* Let  $p$  be odd prime, and let  $m \in \mathbb{Z}$  with  $p \nmid m$ . Applying (2.2) with  $x = -1/2$  and  $y = -4/m$ , we get

$$\sum_{n=0}^{p-1} \left( \sum_{k=0}^n \binom{n}{k} \frac{\binom{2k}{k}}{m^k} \right)^2 \equiv \left( \frac{-1}{p} \right) \pmod{p}. \quad (2.3)$$

By (1.10) with  $x = -1/2$  and  $y = 16/m$ , we have

$$\sum_{n=0}^{p-1} \left( \sum_{k=0}^n \binom{n}{k} \frac{\binom{2k}{k}^2}{m^k} \right)^2 \equiv \left( \frac{-1}{p} \right) \pmod{p}. \quad (2.4)$$

If  $p > 3$ , then by (1.10) with  $x = -1/3, -1/4, -1/6$  we obtain

$$\begin{aligned} \sum_{n=0}^{p-1} \left( \sum_{k=0}^n \binom{n}{k} \frac{\binom{2k}{k} \binom{3k}{k}}{m^k} \right)^2 &\equiv \sum_{n=0}^{p-1} \left( \sum_{k=0}^n \binom{n}{k} \frac{\binom{4k}{2k} \binom{2k}{k}}{m^k} \right)^2 \\ &\equiv \sum_{n=0}^{p-1} \left( \sum_{k=0}^n \binom{n}{k} \frac{\binom{6k}{3k} \binom{3k}{k}}{m^k} \right)^2 \equiv 0 \pmod{p}. \end{aligned} \quad (2.5)$$

**Lemma 2.2.** *Let  $n \in \mathbb{Z}^+$ . Then, for any  $i, j = 0, \dots, n-1$  we have*

$$\sum_{k=0}^{n-1} \binom{k}{i} \binom{k+j}{j} = \frac{(-1)^j n}{i+j+1} \binom{n-1}{i} \binom{-n-1}{j} \quad (2.6)$$

and

$$\begin{aligned} & \sum_{k=0}^{n-1} (2k+1) \binom{k}{i} \binom{k+j}{j} \\ &= (-1)^j \binom{n-1}{i} \binom{-n-1}{j} n \left( \frac{2n}{i+j+2} + \frac{i-j}{(i+j+1)(i+j+2)} \right). \end{aligned} \quad (2.7)$$

*Proof.* For  $m \in \mathbb{Z}^+$  and  $h \in \mathbb{N}$ , we clearly have the known identity

$$\sum_{l=0}^{m-1} \binom{l}{h} = \sum_{l=0}^{m-1} \left( \binom{l+1}{h+1} - \binom{l}{h+1} \right) = \binom{m}{h+1}.$$

Thus

$$\sum_{k=0}^{n-1} \binom{k}{i} \binom{k+j}{j} = \sum_{k=0}^{n-1} \binom{k+j}{i+j} \binom{i+j}{j} = \binom{i+j}{j} \binom{n+j}{i+j+1}.$$

As

$$\frac{i+j+1}{n} \binom{n+j}{i+j+1} = \frac{(n+1) \cdots (n+j)(n-1) \cdots (n-i)}{(i+j)!} = \frac{\binom{n-1}{i} \binom{n+j}{j}}{\binom{i+j}{j}},$$

we have

$$\sum_{k=0}^{n-1} \binom{k}{i} \binom{k+j}{j} = \frac{n}{i+j+1} \binom{n-1}{i} \binom{n+j}{j},$$

which is equivalent to (2.6).

Similarly, we also have

$$\begin{aligned} & \sum_{k=0}^{n-1} (2k+1) \binom{k}{i} \binom{k+j}{j} = \sum_{k=0}^{n-1} (2k+1) \binom{k+j}{i+j} \binom{i+j}{j} \\ &= \binom{i+j}{j} \sum_{k=0}^{n-1} \left( 2(i+j+1) \binom{k+j+1}{i+j+1} - (2j+1) \binom{k+j}{i+j} \right) \\ &= \binom{i+j}{j} \left( 2(i+j+1) \binom{n+j+1}{i+j+2} - (2j+1) \binom{n+j}{i+j+1} \right) \\ &= \left( \frac{2(n+j+1)}{i+j+2} (i+j+1) - (2j+1) \right) \binom{i+j}{j} \binom{n+j}{i+j+1} \\ &= \left( \frac{2n}{i+j+2} (i+j+1) + \frac{(2j+2)(i+j+1) - (2j+1)(i+j+2)}{i+j+2} \right) \\ & \quad \times \frac{n}{i+j+1} \binom{n-1}{i} (-1)^j \binom{-n-1}{j} \\ &= \left( \frac{2n}{i+j+2} + \frac{i-j}{(i+j+1)(i+j+2)} \right) n (-1)^j \binom{n-1}{i} \binom{-n-1}{j}. \end{aligned}$$

This proves (2.7).  $\square$

**Lemma 2.3.** (i) *For any positive integer  $n$ , we have*

$$\sum_{\substack{i, j \in \mathbb{N} \\ i+j=n}} (i-j) \binom{x}{i} \binom{y}{j} = (x-y) \binom{x+y-1}{n-1}. \quad (2.8)$$

(ii) *For any odd prime  $p$  and  $i, j \in \mathbb{N}$  with  $i+j = p-2$ , we have*

$$(i-j) \binom{p-1}{i} \binom{p-1}{j} \equiv j-i-2p \pmod{p^2}. \quad (2.9)$$

*Proof.* (i) In view of the Chu-Vandermonde identity (1.2),

$$\begin{aligned} \sum_{i+j=n} (i-j) \binom{x}{i} \binom{y}{j} &= x \sum_{\substack{i \geq 0 \\ i+j=n}} \binom{x-1}{i-1} \binom{y}{j} - y \sum_{\substack{j \geq 0 \\ i+j=n}} \binom{x}{i} \binom{y-1}{j-1} \\ &= x \binom{x-1+y}{n-1} - y \binom{x+y-1}{n-1} = (x-y) \binom{x+y-1}{n-1}. \end{aligned}$$

This proves (2.8).

(ii) Observe that

$$\begin{aligned} &(i-j) \binom{p-1}{i} \binom{-p-1}{j} \\ &= (i-j) \frac{i+1}{p-i-1} \binom{p-1}{i+1} \binom{-p-1}{j} \\ &= (i-(p-2-i)) \frac{i+1}{p-1-i} \binom{p-1}{j} \binom{-p-1}{j} \\ &\equiv \frac{(p-2(p-1-i))(i+1)}{p-1-i} \binom{-1}{j}^2 = \frac{p(i+1)}{p-1-i} - 2(i+1) \\ &\equiv -p-2i-2 = j-i-2p \pmod{p^2}. \end{aligned}$$

So we have the desired (2.9).  $\square$

*Proof of Theorem 1.2.* Let  $r = \langle x \rangle_p$ . For each  $k = 0, 1, \dots, p-1$ , by Lemma 2.1 we have

$$d_k(r) = \sum_{j=0}^{p-1} \binom{k}{j} \binom{r}{j} 2^j = \sum_{j=0}^r \binom{r}{j} \binom{k}{j} 2^j = \sum_{j=0}^r \binom{r}{j} \binom{k+j}{j} 2^j (-1)^{r-j}.$$

Thus

$$\begin{aligned}
& \sum_{k=0}^{p-1} (-1)^k d_k(x) d_k(r) \\
&= \sum_{k=0}^{p-1} d_k(-1-x) \sum_{j=0}^r \binom{r}{j} \binom{k+j}{j} 2^j (-1)^{r-j} \\
&= \sum_{k=0}^{p-1} \sum_{i=0}^{p-1} \binom{k}{i} \binom{-1-x}{i} 2^i \sum_{j=0}^{p-1} \binom{k+j}{j} \binom{r}{j} 2^j (-1)^{r-j} \\
&= \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \binom{-1-x}{i} \binom{r}{j} 2^{i+j} (-1)^{r-j} \sum_{k=0}^{p-1} \binom{k}{i} \binom{k+j}{j} \\
&= (-1)^r \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \binom{-1-x}{i} \binom{r}{j} 2^{i+j} \frac{p}{i+j+1} \binom{p-1}{i} \binom{-p-1}{j}
\end{aligned}$$

with the help of Lemma 2.2.

For  $i, j \in \{0, \dots, p-1\}$ , we claim that

$$\frac{p}{i+j+1} \binom{p-1}{i} \binom{-p-1}{j} \equiv \frac{(-1)^{i+j} p}{i+j+1} \pmod{p^2}. \quad (2.10)$$

When  $i+j \neq p-1$ , this holds trivially since  $\binom{p-1}{s} \equiv (-1)^s \pmod{p}$  for all  $s = 0, 1, \dots, p-1$ . If  $i+j = p-1$ , then

$$\frac{p}{i+j+1} \binom{p-1}{i} \binom{-p-1}{j} = \binom{p-1}{j} \binom{-p-1}{j} \equiv \binom{-1}{j}^2 = 1 \pmod{p^2}.$$

With the help of (2.10), from the above we deduce that

$$\sum_{k=0}^{p-1} (-1)^k d_k(x) d_k(r) \equiv (-1)^r \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \binom{-1-x}{i} \binom{r}{j} \frac{(-2)^{i+j} p}{i+j+1} \pmod{p^2}. \quad (2.11)$$

As  $-1-x \equiv p-1-r \pmod{p}$ , we have

$$\sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \binom{-1-x}{i} \binom{r}{j} \frac{(-2)^{i+j} p}{i+j+1} \equiv p \Sigma_1 - \Sigma_2 \pmod{p^2}, \quad (2.12)$$

where

$$\Sigma_1 := \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \binom{p-1-r}{i} \binom{r}{j} \frac{(-2)^{i+j}}{i+j+1}$$

and

$$\Sigma_2 := \sum_{i+j=p-1} \left( \binom{p-1-r}{i} - \binom{-1-x}{i} \right) \binom{r}{j} \frac{(-2)^{i+j} p}{i+j+1}.$$

Observe that

$$\begin{aligned} \Sigma_1 &= \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \binom{p-1-r}{i} \binom{r}{j} \int_0^1 (-2u)^{i+j} du \\ &= \int_0^1 \sum_{i=0}^{p-1-r} \binom{p-1-r}{i} (-2u)^i \sum_{j=0}^r \binom{r}{j} (-2u)^j du = \int_0^1 (1-2u)^{p-1} du \\ &= \frac{(1-2u)^p}{-2p} \Big|_{u=0}^1 = \frac{(-1)^p}{-2p} - \frac{1}{-2p} = \frac{1}{p}. \end{aligned}$$

In view of the Chu-Vandermonde identity (1.2), we have

$$\begin{aligned} \Sigma_2 &= (-2)^{p-1} \left( \sum_{i+j=p-1} \binom{p-1-r}{i} \binom{r}{j} - \sum_{i+j=p-1} \binom{-1-x}{i} \binom{r}{j} \right) \\ &= 2^{p-1} \left( \binom{p-1}{p-1} - \binom{-1-x+r}{p-1} \right). \end{aligned}$$

As  $x-r \equiv 0 \pmod{p}$  and

$$\sum_{s=1}^{p-1} \frac{1}{s} = \sum_{s=1}^{(p-1)/2} \left( \frac{1}{s} + \frac{1}{p-s} \right) \equiv 0 \pmod{p},$$

we see that

$$\begin{aligned} \binom{-1-x+r}{p-1} &= \prod_{s=1}^{p-1} \frac{r-x-s}{s} = \prod_{s=1}^{p-1} \left( 1 + \frac{x-r}{s} \right) \\ &\equiv 1 + \sum_{s=1}^{p-1} \frac{x-r}{s} \equiv 1 \pmod{p^2}. \end{aligned}$$

Therefore  $\Sigma_2 \equiv 0 \pmod{p^2}$ . Combining these with (2.11) and (2.12), we find that

$$\sum_{k=0}^{p-1} (-1)^k d_k(x) d_k(r) \equiv (-1)^r \pmod{p^2}. \quad (2.13)$$

In particular,

$$\sum_{k=0}^{p-1} (-1)^k d_k(r)^2 \equiv (-1)^r \pmod{p^2}. \quad (2.14)$$

As  $(d_k(x) - d_k(r))^2 \equiv 0 \pmod{p^2}$  for all  $k = 0, \dots, p-1$ , by (1.7), (2.13) and (2.14) we finally obtain

$$\begin{aligned} \sum_{k=0}^{p-1} (-1)^k d_k(x)^2 &= \sum_{k=0}^{p-1} d_k(x) d_k(-1-x) \\ &\equiv \sum_{k=0}^{p-1} (-1)^k (2d_k(x) d_k(r) - d_k(r)^2) \\ &\equiv 2(-1)^r - (-1)^r = (-1)^r \pmod{p^2}. \end{aligned}$$

This proves (1.11).

Similar to (2.11), we have

$$\sum_{k=0}^{p-1} d_k(x) d_k(r) \equiv (-1)^r \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \binom{x}{i} \binom{r}{j} \frac{(-2)^{i+j} p}{i+j+1} \pmod{p^2}. \quad (2.15)$$

As  $x \equiv r \pmod{p}$ ,

$$\sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \binom{x}{i} \binom{r}{j} \frac{(-2)^{i+j} p}{i+j+1} \equiv p\sigma_1 + \sigma_2 \pmod{p^2}, \quad (2.16)$$

where

$$\begin{aligned} \sigma_1 &= \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \binom{r}{i} \binom{r}{j} \frac{(-2)^{i+j}}{i+j+1} \\ &= \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \binom{r}{i} \binom{r}{j} \int_0^1 (-2v)^{i+j} dv = \int_0^1 (1-2v)^r (1-2v)^r dv \\ &= \frac{(1-2v)^{2r+1}}{-2(2r+1)} \Big|_{v=0}^1 = \frac{-1}{-2(2r+1)} - \frac{1}{-2(2r+1)} = \frac{1}{2r+1} \end{aligned}$$

and

$$\begin{aligned} \sigma_2 &= \sum_{i+j=p-1} \left( \binom{x}{i} - \binom{r}{i} \right) \binom{r}{j} \frac{(-2)^{i+j} p}{i+j+1} \\ &= 2^{p-1} \left( \sum_{i+j=p-1} \binom{x}{i} \binom{r}{j} - \sum_{i+j=p-1} \binom{r}{i} \binom{r}{j} \right) \\ &= 2^{p-1} \left( \binom{x+r}{p-1} - \binom{2r}{p-1} \right). \end{aligned}$$

Write  $x = r + pt$  with  $t$  a  $p$ -adic integer. If  $2r = p - 1$  (i.e.,  $x \equiv -1/2 \pmod{p}$ ), then

$$\begin{aligned} \binom{x+r}{p-1} - \binom{2r}{p-1} &= \binom{p(t+1)-1}{p-1} - 1 = \prod_{s=1}^{p-1} \left(1 - \frac{p(t+1)}{s}\right) - 1 \\ &\equiv -p(t+1) \sum_{s=1}^{p-1} \frac{1}{s} \equiv 0 \pmod{p^2}. \end{aligned}$$

If  $2r < p - 1$ , then

$$\begin{aligned} \binom{x+r}{p-1} - \binom{2r}{p-1} &= \binom{pt+2r}{p-1} \\ &= \frac{(pt+2r) \cdots (pt+1)pt(pt-1) \cdots (pt-(p-1-2r))}{(p-1)!(pt-(p-1-2r))} \\ &\equiv \frac{pt}{2r+1} \cdot \frac{(2r)!(p-1-2r)!}{(p-1)!} = \frac{pt}{(2r+1)\binom{p-1}{2r}} \equiv \frac{pt}{2r+1} \pmod{p^2}. \end{aligned}$$

If  $2r > p - 1$ , then  $2r - p \in \{0, \dots, p-1\}$  and

$$\begin{aligned} \binom{x+r}{p-1} - \binom{2r}{p-1} &= \binom{pt+2r}{p-1} - \binom{2r}{p-1} \\ &= \frac{(p(t+1)+2r-p) \cdots p(t+1) \cdots (p(t+1)-(2p-2r-1))}{(p-1)!(p(t+1)-(2p-2r-1))} \\ &\quad - \frac{(p+2r-p) \cdots (p+1)p(p-1) \cdots (p-(2p-2r-1))}{(p-1)!(p-(2p-2r-1))} \\ &\equiv -\frac{p(t+1)}{2r+1} \cdot \frac{(2r-p)!(2p-2r-1)!}{(p-1)!} + \frac{p}{2r+1} \cdot \frac{(2r-p)!(2p-2r-1)!}{(p-1)!} \\ &= -\frac{pt}{2r+1} \cdot \frac{1}{\binom{p-1}{2r-p}} \equiv \frac{pt}{2r+1} \pmod{p^2}. \end{aligned}$$

Therefore

$$\sigma_2 \equiv \begin{cases} 0 \pmod{p^2} & \text{if } r = (p-1)/2, \\ pt/(2r+1) \pmod{p^2} & \text{otherwise.} \end{cases}$$

Combining (2.15), (2.16) and our results on  $\sigma_1$  and  $\sigma_2$  modulo  $p^2$ , we obtain

$$\sum_{k=0}^{p-1} d_k(x)d_k(r) \equiv \begin{cases} (-1)^r p/(2r+1) = (-1)^r \pmod{p^2} & \text{if } r = (p-1)/2, \\ (-1)^r p(t+1)/(2r+1) \pmod{p^2} & \text{otherwise.} \end{cases} \quad (2.17)$$

In particular,

$$\sum_{k=0}^{p-1} d_k(r)^2 \equiv \begin{cases} (-1)^r \pmod{p^2} & \text{if } r = (p-1)/2, \\ (-1)^r p/(2r+1) \pmod{p^2} & \text{otherwise.} \end{cases} \quad (2.18)$$

As  $(d_k(x) - d_k(r))^2 \equiv 0 \pmod{p^2}$ , from (2.17) and (2.18) we get

$$\begin{aligned} \sum_{k=0}^{p-1} d_k(x)^2 &\equiv 2 \sum_{k=0}^{p-1} d_k(x) d_k(r) - \sum_{k=0}^{p-1} d_k(r)^2 \\ &\equiv \begin{cases} (-1)^r \pmod{p^2} & \text{if } r = (p-1)/2, \\ (-1)^r (p+2t)/(2r+1) \pmod{p^2} & \text{otherwise.} \end{cases} \end{aligned}$$

Combining this with (1.7) we obtain the desired (1.12).

Now we deduce (1.13). If  $x \equiv 0 \pmod{p}$ , then

$$\binom{x}{j} = \frac{x}{j} \prod_{0 < i < j} \frac{x-i}{i} \equiv \frac{x}{j} (-1)^{j-1} \pmod{p^2}$$

for all  $j = 0, \dots, p-1$  and hence  $d_k(x) \equiv 1 \pmod{p}$  for all  $k = 0, \dots, p-1$ , thus

$$\begin{aligned} \sum_{k=0}^{p-1} (2k+1) d_k(x)^2 &= \sum_{k=0}^{p-1} (2k+1) (1 + 2(d_k(x) - 1) + (d_k(x) - 1)^2) \\ &\equiv \sum_{k=0}^{p-1} (2k+1) (2d_k(x) - 1) = 2 \sum_{k=0}^{p-1} (2k+1) (d_k(x) - 1) + p^2 \\ &\equiv 2 \sum_{k=1}^{p-1} (2(k+1) - 1) \sum_{j=1}^k \binom{k}{j} \binom{x}{j} 2^j \\ &\equiv 2x \sum_{j=1}^{p-1} \frac{(-1)^{j-1}}{j} 2^j \sum_{k=j}^{p-1} \left( 2(j+1) \binom{k+1}{j+1} - \binom{k}{j} \right) \\ &= 2x \sum_{j=1}^{p-1} \frac{(-1)^{j-1} 2^j}{j} \left( (2j+2) \binom{p+1}{j+2} - \binom{p}{j+1} \right) \\ &\equiv 2x \sum_{j=p-2}^{p-1} \frac{(-1)^{j-1} 2^j}{j} \left( (2j+2) \binom{p+1}{j+2} - \binom{p}{j+1} \right) \pmod{p^2} \end{aligned}$$

and hence

$$\begin{aligned} &\sum_{k=0}^{p-1} (2k+1) d_k(x)^2 \\ &\equiv 2x \left( \frac{2^{p-2}}{p-2} \left( 2(p-1) \binom{p+1}{p} - p \right) - \frac{2^{p-1}}{p-1} (2p-1) \right) \equiv -x \pmod{p^2}. \end{aligned}$$

When  $x \equiv -1 \pmod{p}$ , we have  $x' := -1 - x \equiv 0 \pmod{p}$  and hence

$$\sum_{k=0}^{p-1} (2k+1) d_k(x)^2 = \sum_{k=0}^{p-1} (2k+1) ((-1)^k d_k(x'))^2 \equiv -x' = x+1 \pmod{p^2}$$



with the help of (1.7).

Below we assume  $x \not\equiv 0, -1 \pmod{p}$  (i.e.,  $0 < r = \langle x \rangle_p < p-1$ ). Similar to (2.11) and (2.15), with the help of (2.7) we have

$$\begin{aligned} \sum_{k=0}^{p-1} (2k+1)d_k(x)d_k(r) &= (-1)^r \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \binom{x}{i} \binom{r}{j} 2^{i+j} \binom{p-1}{i} \binom{-p-1}{j} \\ &\quad \times p \left( \frac{2p}{i+j+2} + \frac{i-j}{(i+j+1)(i+j+2)} \right). \end{aligned}$$

Note that

$$\binom{x}{i} \binom{r}{j} 2^{i+j} \binom{p-1}{i} \binom{-p-1}{j} \equiv \binom{r}{i} \binom{r}{j} (-2)^{i+j} \pmod{p}.$$

for any  $i, j \in \{0, \dots, p-1\}$ . Using the symmetry of  $i$  and  $j$  in  $\binom{r}{i} \binom{r}{j} (-2)^{i+j}$ , we deduce that

$$\sum_{k=0}^{p-1} (2k+1)d_k(x)d_k(r) \equiv (-1)^r (2^{p-1}w_{p-1} + 2^{p-2}w_{p-2}) \pmod{p^2}, \quad (2.19)$$

where

$$w_n := \sum_{i+j=n} \binom{x}{i} \binom{r}{j} \binom{p-1}{i} \binom{-p-1}{j} p \left( \frac{2p}{i+j+2} + \frac{i-j}{(i+j+1)(i+j+2)} \right).$$

If  $i+j = p-1$ , then

$$\binom{p-1}{i} \binom{-p-1}{j} = \binom{p-1}{j} \binom{-p-1}{j} \equiv \binom{-1}{j}^2 = 1 \pmod{p^2}.$$

By Lemma 2.3,

$$\begin{aligned} \sum_{i+j=p-1} (i-j) \binom{x}{i} \binom{r}{j} &= (x-r) \binom{x+r-1}{p-2} = \frac{(x-r)(p-1)}{x+r} \binom{x+r}{p-1} \\ &\equiv \frac{r-x}{2r} \binom{2r}{p-1} \equiv \begin{cases} x-r \pmod{p^2} & \text{if } 2r = p-1, \\ 0 \pmod{p^2} & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore

$$\begin{aligned} w_{p-1} &\equiv \sum_{i+j=p-1} \binom{x}{i} \binom{r}{j} \binom{p-1}{i} \binom{-p-1}{j} \frac{i-j}{p+1} \\ &\equiv \frac{1}{p+1} \sum_{i+j=p-1} (i-j) \binom{x}{i} \binom{r}{j} \equiv \begin{cases} x-r \pmod{p^2} & \text{if } 2r = p-1, \\ 0 \pmod{p^2} & \text{otherwise.} \end{cases} \end{aligned}$$

Lemma 2.3 also implies that

$$\begin{aligned} \sum_{i+j=p-2} (i-j) \binom{x}{i} \binom{r}{j} &= (x-r) \binom{x+r-1}{p-3} = \frac{(x-r)(p-2)}{x+r} \binom{x+r}{p-2} \\ &\equiv \frac{r-x}{r} \binom{2r}{p-2} \equiv \begin{cases} 2(r-x) \pmod{p^2} & \text{if } 2r = p-1, \\ 0 \pmod{p^2} & \text{otherwise.} \end{cases} \end{aligned}$$

In view of this and (2.9), we find that

$$\begin{aligned} w_{p-2} &\equiv \sum_{i+j=p-2} \binom{x}{i} \binom{r}{j} \binom{p-1}{i} \binom{-p-1}{j} \left(2p + \frac{i-j}{p-1}\right) \\ &\equiv 2p \sum_{i+j=p-2} \binom{x}{i} \binom{r}{j} (-1)^{i+j} + \sum_{i+j=p-2} \binom{x}{i} \binom{r}{j} \frac{j-i-2p}{p-1} \\ &\equiv \frac{1}{1-p} \sum_{i+j=p-2} (i-j) \binom{x}{i} \binom{r}{j} \equiv \begin{cases} 2(r-x) \pmod{p^2} & \text{if } 2r = p-1, \\ 0 \pmod{p^2} & \text{otherwise.} \end{cases} \end{aligned}$$

Combining (2.19) with the relation  $w_{p-1} + w_{p-2}/2 \equiv 0 \pmod{p^2}$ , we get

$$\sum_{k=0}^{p-1} (2k+1) d_k(x) d_k(r) \equiv 0 \pmod{p^2}. \quad (2.20)$$

In particular,

$$\sum_{k=0}^{p-1} (2k+1) d_k(r)^2 \equiv 0 \pmod{p^2}. \quad (2.21)$$

Since  $(d_k(x) - d_k(r))^2 \equiv 0 \pmod{p^2}$ , by (2.20) and (2.21) we finally obtain

$$\sum_{k=0}^{p-1} (2k+1) d_k(x)^2 \equiv 2 \sum_{k=0}^{p-1} (2k+1) d_k(x) d_k(r) - \sum_{k=0}^{p-1} (2k+1) d_k(r)^2 \equiv 0 \pmod{p^2}.$$

The proof of Theorem 1.2 is now complete.  $\square$

*Proof of Corollary 1.1.* Clearly  $\langle -1/4 \rangle_p$  is  $(p-1)/4$  or  $(3p-1)/4$  according as  $p \equiv 1 \pmod{4}$  or not. If  $p > 3$ , then  $\langle -1/3 \rangle_p$  is  $(p-1)/3$  or  $(2p-1)/3$  according as  $p \equiv 1 \pmod{3}$  or not, and  $\langle -1/6 \rangle_p$  is  $(p-1)/6$  or  $(5p-1)/6$  according as  $p \equiv 1 \pmod{6}$  or not. Thus, by applying Theorem 1.2 with  $x = -1/4, -1/3, -1/6$  we immediately obtain the desired congruences. This concludes the proof.  $\square$

### 3. PROOFS OF THEOREM 1.3 AND COROLLARY 1.2

*Proof of Theorem 1.3.* Observe that

$$(-1)^{k+m} \binom{y}{k+m} = \binom{-y+k+m-1}{k+m} = \sum_{j=0}^k \binom{k}{k-j} \binom{m-1-y}{j+m}$$

for all  $k = 0, \dots, n$  by the Chu-Vandermonde identity (1.2). So

$$\begin{aligned}
 & \sum_{k=0}^n (-1)^{k+m} \binom{x}{k} \binom{y}{k+m} \binom{z}{n-k} \\
 &= \sum_{k=0}^n \binom{x}{k} \binom{z}{n-k} \sum_{j=0}^n \binom{k}{j} \binom{m-1-y}{j+m} \\
 &= \sum_{j=0}^n \binom{x}{j} \binom{m-1-y}{j+m} \sum_{k=j}^n \binom{x-j}{k-j} \binom{z}{n-k} \\
 &= \sum_{j=0}^n \binom{x}{j} \binom{m-1-y}{j+m} \binom{x-j+z}{n-j}
 \end{aligned}$$

with the help of the Chu-Vandermonde identity. As

$$\binom{x-j+z}{n-j} = (-1)^{n-j} \binom{-x-z+n-1}{n-j},$$

we obtain (1.15) from the above.

Let  $d \in \mathbb{N}$ . Applying (1.15) with  $z = n + d$  we get

$$\begin{aligned}
 & \sum_{k=0}^n (-1)^{m+n-k} \binom{n+d}{k+d} \binom{x}{k} \binom{y}{k+m} \\
 &= \sum_{k=0}^n (-1)^k \binom{x}{k} \binom{m-1-y}{k+m} \binom{-x-d-1}{n-k}.
 \end{aligned} \tag{3.1}$$

When  $m = 0$  and  $x + y = -1$ , this reduces to (1.16).

As  $x$  and  $y$  on the left-hand side of (1.16) are symmetric, from (1.16) we know that  $P(z) - Q(z) = 0$  for all  $z = 0, -1, -2, \dots$ , where  $P(z)$  and  $Q(z)$  denote the left-hand side and the right-hand side of (1.13) respectively. Thus the polynomials  $P(z)$  and  $Q(z)$  are identical. This proves (1.17).  $\square$

*Proof of Corollary 1.2.* Let  $d \in \mathbb{N}$ . Observe that

$$\begin{aligned}
 \sum_{k=0}^n \binom{n+d}{k+d} \binom{x}{k} \binom{x+k}{k} &= \sum_{k=0}^{m-1} (-1)^k \binom{n+d}{k+d} \binom{x}{k} \binom{-1-x}{k} \\
 &= \sum_{k=0}^x (-1)^k \binom{-1-x}{k} \binom{n+d}{k+d} \binom{x}{x-k}.
 \end{aligned}$$

By Theorem 1.3, we have

$$\begin{aligned}
& \sum_{k=0}^x (-1)^k \binom{-1-x}{k} \binom{n+d}{k+d} \binom{x}{x-k} \\
&= (-1)^{d+x} \sum_{k=0}^x (-1)^k \binom{-1-x}{k} \binom{d-1-(n+d)}{k+d} \binom{x-(-1-x+x+1)}{x-k} \\
&= (-1)^x \sum_{k=0}^x (-1)^{k+d} \binom{x}{k} \binom{-1-x}{k} \binom{-n-1}{k+d} \\
&= (-1)^x \sum_{k=0}^{m-1} \binom{x}{k} \binom{-1-x}{k} \binom{n+k+d}{k+d}.
\end{aligned}$$

Thus (1.18) holds. Putting  $d = 0$  in (1.18) we get (1.19).  $\square$

#### 4. SOME LEMMAS

Let  $p > 3$  be a prime and let  $a_0, a_1, \dots$  be  $p$ -adic integers. In 2013 the author [S13, Theorem 1.4] proved the following congruences:

$$\begin{aligned}
\sum_{k=0}^{p-1} \frac{\binom{3k}{k} \binom{2k}{k}}{27^k} a_k &\equiv \left(\frac{p}{3}\right) \sum_{k=0}^{p-1} \frac{\binom{3k}{k} \binom{2k}{k}}{27^k} a_k^* \pmod{p^2}, \\
\sum_{k=0}^{p-1} \frac{\binom{4k}{2k} \binom{2k}{k}}{64^k} a_k &\equiv \left(\frac{-2}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{4k}{2k} \binom{2k}{k}}{64^k} a_k^* \pmod{p^2}, \\
\sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{432^k} a_k &\equiv \left(\frac{-1}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{432^k} a_k^* \pmod{p^2}.
\end{aligned}$$

Note that

$$\binom{-1/3}{k} \binom{-2/3}{k} = \frac{\binom{2k}{k} \binom{3k}{k}}{27^k}, \quad \binom{-1/4}{k} \binom{-3/4}{k} = \frac{\binom{4k}{2k} \binom{2k}{k}}{64^k}$$

and

$$\binom{-1/6}{k} \binom{-5/6}{k} = \frac{\binom{6k}{3k} \binom{3k}{k}}{432^k}.$$

In 2014 Z.-H. Sun [S2, Theorem 2.4] obtained the following extension of the above result: For any  $p$ -adic integer  $x$  we have

$$\sum_{k=0}^{p-1} \binom{x}{k} \binom{-1-x}{k} a_k \equiv (-1)^{\langle x \rangle_p} \sum_{k=0}^{p-1} \binom{x}{k} \binom{-1-x}{k} a_k^*. \quad (4.1)$$

**Lemma 4.1.** *Let  $p$  be an odd prime and let  $k \in \{0, 1, \dots, p-1\}$ . For any  $p$ -adic integer  $x$ , we have*

$$s_k(x) \equiv (-1)^{\langle x \rangle_p} \sum_{j=0}^{p-1} \binom{x}{j} \binom{-1-x}{j} \binom{k+j}{j} \pmod{p^2}. \quad (4.2)$$

*Proof.* Clearly,

$$s_k(x) = \sum_{j=0}^{p-1} \binom{x}{j} \binom{-1-x}{j} a_j$$

where  $a_j = (-1)^j \binom{k}{j}$  for  $j = 0, 1, \dots$ . Note that

$$a_j^* = \sum_{i=0}^j \binom{j}{i} (-1)^i a_i = \sum_{i=0}^j \binom{j}{j-i} \binom{k}{i} = \binom{j+k}{j}$$

with the help of the Chu-Vandermonde identity (1.2). So we may apply (4.1) to obtain (4.2).  $\square$

**Lemma 4.2.** *Let  $n$  be any positive integer. Then*

$$\begin{aligned} & \sum_{k=0}^{n-1} s_k(x) \sum_{j=0}^{n-1} \binom{x}{j} \binom{-1-x}{j} \binom{k+j}{j} \\ &= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \binom{x}{i} \binom{x+i}{i} \binom{x}{j} \binom{x+j}{j} \binom{n-1}{i} \binom{-n-1}{j} \frac{n}{i+j+1}. \end{aligned} \quad (4.3)$$

and

$$\begin{aligned} & \frac{1}{n} \sum_{k=0}^{n-1} (2k+1) s_k(x) \sum_{j=0}^{n-1} \binom{x}{j} \binom{-1-x}{j} \binom{k+j}{j} \\ &= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \binom{x}{i} \binom{x+i}{i} \binom{x}{j} \binom{x+j}{j} \binom{n-1}{i} \binom{-n-1}{j} \\ & \quad \times \left( \frac{2n}{i+j+2} + \frac{i-j}{(i+j+1)(i+j+2)} \right). \end{aligned} \quad (4.4)$$

*Proof.* Observe that

$$\begin{aligned} & \sum_{k=0}^{n-1} s_k(x) \sum_{j=0}^{n-1} \binom{x}{j} \binom{-1-x}{j} \binom{k+j}{j} \\ &= \sum_{k=0}^{n-1} \sum_{i=0}^{n-1} (-1)^i \binom{k}{i} \binom{x}{i} \binom{-1-x}{i} \sum_{j=0}^{n-1} \binom{x}{j} \binom{-x-1}{j} \binom{k+j}{j} \\ &= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (-1)^i \binom{x}{i} \binom{-1-x}{i} \binom{x}{j} \binom{-1-x}{j} \sum_{k=0}^{n-1} \binom{k}{i} \binom{k+j}{j}. \end{aligned}$$

Combining this with (2.6), we obtain

$$\begin{aligned}
& \sum_{k=0}^{n-1} s_k(x) \sum_{j=0}^{p-1} \binom{x}{j} \binom{-1-x}{j} \binom{k+j}{j} \\
&= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (-1)^i \binom{x}{i} \binom{-1-x}{i} \binom{x}{j} \binom{-1-x}{j} \frac{n}{i+j+1} (-1)^j \binom{n-1}{i} \binom{-n-1}{j} \\
&= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \binom{x}{i} \binom{x+i}{i} \binom{x}{j} \binom{x+j}{j} \frac{n}{i+j+1} \binom{n-1}{i} \binom{-n-1}{j}.
\end{aligned}$$

This proves (4.3).

Similar to the above, we have

$$\begin{aligned}
& \sum_{k=0}^{n-1} (2k+1) s_k(x) \sum_{j=0}^{n-1} \binom{x}{j} \binom{-1-x}{j} \binom{k+j}{j} \\
&= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (-1)^i \binom{x}{i} \binom{-1-x}{i} \binom{x}{j} \binom{-1-x}{j} \sum_{k=0}^{n-1} (2k+1) \binom{k}{i} \binom{k+j}{j}.
\end{aligned}$$

Combining this with (2.7) we obtain the desired (4.4).  $\square$

As usual, we set

$$H_n := \sum_{0 < k \leq n} \frac{1}{k} \quad \text{and} \quad H_n^{(2)} := \sum_{0 < k \leq n} \frac{1}{k^2} \quad \text{for } n = 0, 1, 2, \dots$$

A classical result of Wolstenholme [W] states that

$$H_{p-1} \equiv 0 \pmod{p^2} \quad \text{and} \quad H_{p-1}^{(2)} \equiv 0 \pmod{p}$$

for any prime  $p > 3$ .

**Lemma 4.3.** *Let  $p > 3$  be a prime and let  $i, j \in \{0, \dots, p-1\}$ . Then*

$$\begin{aligned}
& \frac{p}{i+j+1} \binom{p-1}{i} \binom{-p-1}{j} \\
&\equiv \begin{cases} \frac{(-1)^{i+j}}{i+j+1} (p - p^2(H_i - H_j)) \pmod{p^3} & \text{if } i+j \neq p-1, \\ 1 + p^2 H_i^{(2)} \equiv 1 - p^2 H_j^{(2)} \pmod{p^3} & \text{if } i+j = p-1. \end{cases} \quad (4.5)
\end{aligned}$$

*Proof.* Note that

$$\begin{aligned}
(-1)^{i+j} \binom{p-1}{i} \binom{-p-1}{j} &= \prod_{0 < s \leq i} \left(1 - \frac{p}{s}\right) \times \prod_{0 < t \leq j} \left(1 - \frac{p}{t}\right) \\
&\equiv (1 - pH_i)(1 - pH_j) \equiv (1 - p(H_i - H_j)) \pmod{p^2}.
\end{aligned}$$

So (4.5) holds in the case  $i + j \neq p - 1$ . When  $i + j = p - 1$ , we have

$$\begin{aligned} & \frac{p}{i+j+1} \binom{p-1}{i} \binom{-p-1}{j} \\ &= \binom{p-1}{j} \binom{-p-1}{j} = \prod_{0 < r \leq j} \left( \frac{p-r}{r} \cdot \frac{-p-r}{r} \right) = \prod_{0 < r \leq j} \left( 1 - \frac{p^2}{r^2} \right) \\ &\equiv 1 - p^2 H_j^{(2)} = 1 - p^2 \left( H_{p-1}^{(2)} - \sum_{0 < r \leq i} \frac{1}{(p-r)^2} \right) \equiv 1 + p^2 H_i^{(2)} \pmod{p^3}. \end{aligned}$$

This concludes the proof.  $\square$

**Lemma 4.4.** *For any  $n \in \mathbb{N}$  we have the identity*

$$\sum_{\substack{i, j \in \mathbb{N} \\ i+j=n}} \binom{x}{i} \binom{x+i}{i} \binom{x}{j} \binom{x+j}{j} = \sum_{k=0}^n \binom{x}{k} \binom{x+k}{k} \binom{2k}{k} \binom{k}{n-k}. \quad (4.6)$$

*Proof.* Let  $u_n$  denote the left-hand side or the right-hand side of (4.6). It is easy to see that  $u_0 = 1$  and  $u_1 = 2x(x+1)$ . Applying the Zeilberger algorithm (cf. [PWZ]) we find that

$$(n+1)(n-2x)(n+2x+2)u_n + (2n+3)(n^2+3n+2-2x^2-2x)u_{n+1} + (n+2)^3 u_{n+2} = 0$$

for all  $n = 0, 1, 2, \dots$ . Thus (4.6) holds by induction on  $n$ .  $\square$

**Lemma 4.5.** *Let  $p$  be an odd prime. Then we have*

$$\begin{aligned} & \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \binom{x}{i} \binom{-1-x}{i} \binom{x}{j} \binom{-1-x}{j} \frac{p}{i+j+1} \\ &\equiv \sum_{k=0}^{p-1} \binom{x}{k} \binom{-1-x}{k} \frac{p}{2k+1} \pmod{p^3}. \end{aligned} \quad (4.7)$$

*Proof.* Clearly  $\binom{x}{k} \binom{-1-x}{k} = \binom{x}{k} \binom{x+k}{k} (-1)^k$  for all  $k \in \mathbb{N}$ . By [Su14a, Theorem 3.2], for some  $f(x, z) \in \mathbb{Z}_p[x, z]$  we have

$$\left( \sum_{k=0}^{p-1} \binom{x}{k} \binom{x+k}{k} z^k \right)^2 = \sum_{k=0}^{p-1} \binom{x}{k} \binom{x+k}{k} \binom{2k}{k} (z(z+1))^k + p^2 f(x, z). \quad (4.8)$$

Obviously, the degree of  $f(x, z)$  in  $z$  is at most  $2(p-1)$ . Note that the coefficient of  $z^{p-1}$  in

$$\sum_{k=0}^{p-1} \binom{x}{k} \binom{x+k}{k} \binom{2k}{k} (z(z+1))^k$$

coincides with

$$\sum_{k=0}^{p-1} \binom{x}{k} \binom{x+k}{k} \binom{2k}{k} \binom{k}{p-1-k},$$

which is equal to

$$\sum_{i+j=p-1} \binom{x}{i} \binom{x+i}{i} \binom{x}{j} \binom{x+j}{j}$$

by Lemma 4.3. So the coefficient of  $z^{p-1}$  in the polynomial  $f(x, z)$  is zero. For  $k = 0, \dots, 2p-2$  with  $k \neq p-1$ , clearly  $\int_{-1}^0 z^k dz = (-1)^k / (k+1) \in \mathbb{Z}_p$ . Therefore, (4.8) implies that

$$\begin{aligned} \Delta_p(x) &:= \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \binom{x}{i} \binom{x+i}{i} \binom{x}{j} \binom{x+j}{j} \int_{-1}^0 z^{i+j} dz \\ &\quad - \sum_{k=0}^{p-1} \binom{x}{k} \binom{x+k}{k} \binom{2k}{k} \int_{-1}^0 (z(z+1))^k dz \end{aligned}$$

belongs to  $p^2 \mathbb{Z}_p[x]$ . Observe that

$$\begin{aligned} \Delta_p(x) &= \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \binom{x}{i} \binom{x+i}{i} \binom{x}{j} \binom{x+j}{j} \frac{(-1)^{i+j}}{i+j+1} \\ &\quad - \sum_{k=0}^{p-1} \binom{x}{k} \binom{x+k}{k} \binom{2k}{k} \int_0^1 (-z(1-z))^k dz. \end{aligned}$$

It is well-known that

$$\int_0^1 z^{a-1} (1-z)^{b-1} dz = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

for any  $a > 0$  and  $b > 0$ . Thus

$$\int_0^1 z^k (1-z)^k dz = \frac{\Gamma(k+1)\Gamma(k+1)}{\Gamma(2k+2)} = \frac{k!k!}{(2k+1)!} = \frac{1}{(2k+1)\binom{2k}{k}}$$

for all  $k = 0, \dots, p-1$ , and hence  $\Delta_p(x)$  equals

$$\sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \binom{x}{i} \binom{-1-x}{i} \binom{x}{j} \binom{-1-x}{j} \frac{1}{i+j+1} - \sum_{k=0}^{p-1} \binom{x}{k} \binom{-1-x}{k} \frac{1}{2k+1}.$$

So the desired (4.7) follows.  $\square$



## 5. PROOF OF THEOREM 1.4

Let us first prove an auxiliary theorem.

**Theorem 5.1.** *Let  $p > 3$  be a prime. Then we have*

$$\sum_{k=0}^{p-1} s_k(x) \sum_{j=0}^{p-1} \binom{x}{j} \binom{-1-x}{j} \binom{k+j}{j} \equiv \sum_{k=0}^{p-1} \binom{x}{k} \binom{-1-x}{k} \frac{p}{2k+1} \pmod{p^3}. \quad (5.1)$$

*Proof.* Note that

$$H_{(p-1)/2}^{(2)} = \frac{1}{2} \sum_{k=1}^{(p-1)/2} \left( \frac{1}{k^2} + \frac{1}{(p-k)^2} \right) = \frac{1}{2} H_{p-1}^{(2)} \equiv 0 \pmod{p}.$$

In view of this and Lemma 4.3, we have

$$\begin{aligned} & \sum_{i+j=p-1} \binom{x}{i} \binom{x+i}{i} \binom{x}{j} \binom{x+j}{j} \frac{p}{i+j+1} \binom{p-1}{i} \binom{-p-1}{j} \\ & - \sum_{i+j=p-1} \binom{x}{i} \binom{x+i}{i} \binom{x}{j} \binom{x+j}{j} \\ & \equiv \sum_{i+j=p-1} \binom{x}{i} \binom{x+i}{i} \binom{x}{j} \binom{x+j}{j} p^2 H_i^{(2)} \\ & \equiv \sum_{\substack{0 \leq i < j < p \\ i+j=p-1}} \binom{x}{i} \binom{x+i}{i} \binom{x}{j} \binom{x+j}{j} p^2 (H_i^{(2)} + H_j^{(2)}) \equiv 0 \pmod{p^3}. \end{aligned}$$

By Lemma 4.3, we also have

$$\begin{aligned} & \sum_{\substack{i,j \in \{0, \dots, p-1\} \\ i+j \neq p-1}} \binom{x}{i} \binom{x+i}{i} \binom{x}{j} \binom{x+j}{j} \frac{p}{i+j+1} \binom{p-1}{i} \binom{-p-1}{j} \\ & \equiv \sum_{\substack{i,j \in \{0, \dots, p-1\} \\ i+j \neq p-1}} \binom{x}{i} \binom{x+i}{i} \binom{x}{j} \binom{x+j}{j} \frac{(-1)^{i+j}}{i+j+1} (p - p^2 (H_i - H_j)) \\ & = \sum_{\substack{i,j \in \{0, \dots, p-1\} \\ i+j \neq p-1}} \binom{x}{i} \binom{x+i}{i} \binom{x}{j} \binom{x+j}{j} \frac{(-1)^{i+j} p}{i+j+1} \pmod{p^3}. \end{aligned}$$

Therefore

$$\begin{aligned}
& \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \binom{x}{i} \binom{x+i}{i} \binom{x}{j} \binom{x+j}{j} \frac{p}{i+j+1} \binom{p-1}{i} \binom{-p-1}{j} \\
& \equiv \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \binom{x}{i} \binom{x+i}{i} \binom{x}{j} \binom{x+j}{j} \frac{(-1)^{i+j} p}{i+j+1} \\
& = \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \binom{x}{i} \binom{-1-x}{i} \binom{x}{j} \binom{-1-x}{j} \frac{p}{i+j+1} \\
& \equiv \sum_{k=0}^{p-1} \binom{x}{k} \binom{-1-x}{k} \frac{p}{2k+1} \pmod{p^3}
\end{aligned}$$

with the help of Lemma 4.5. Combining this with (4.3), we obtain (5.1).  $\square$

*Proof of Theorem 1.4.* (i) By (1.19) and (5.1), if  $x \in \{0, \dots, p-1\}$  then

$$\begin{aligned}
\sum_{k=0}^{p-1} s_k(x)^2 &= (-1)^x \sum_{k=0}^{p-1} s_k(x) \sum_{j=0}^{p-1} \binom{x}{j} \binom{-1-x}{j} \binom{k+j}{j} \\
&\equiv (-1)^x \sum_{k=0}^{p-1} \binom{x}{k} \binom{-1-x}{k} \frac{p}{2k+1} \pmod{p^3}.
\end{aligned}$$

By Lemma 4.1, we have

$$s_k(x)^2 \equiv (-1)^{\langle x \rangle_p} s_k(x) \sum_{j=0}^{p-1} \binom{x}{j} \binom{-1-x}{j} \binom{k+j}{j} \pmod{p^2}$$

for all  $k = 0, \dots, p-1$ , and hence

$$\begin{aligned}
\sum_{k=0}^{p-1} s_k(x)^2 &\equiv (-1)^{\langle x \rangle_p} \sum_{k=0}^{p-1} s_k(x) \sum_{j=0}^{p-1} \binom{x}{j} \binom{-1-x}{j} \binom{k+j}{j} \\
&\equiv (-1)^{\langle x \rangle_p} \sum_{k=0}^{p-1} \binom{x}{k} \binom{-1-x}{k} \frac{p}{2k+1} \pmod{p^2}
\end{aligned}$$

with the help of (5.1). If  $x \not\equiv -1/2 \pmod{p}$ , then

$$\sum_{k=0}^{p-1} \binom{x}{k} \binom{-1-x}{k} \frac{p}{2k+1} \equiv \frac{p + 2(x - \langle x \rangle_p)}{2x + 1} \pmod{p^3} \quad (5.2)$$

by [S2, (1.14)], and hence (1.20) and (1.21) hold. Note that (1.21) in the case  $x = (p-1)/2$  was proved in [LOS].

(ii) In view of (4.2) and (4.4), we have reduced (1.22) to the congruence

$$\begin{aligned} & \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \binom{x}{i} \binom{x+i}{i} \binom{x}{j} \binom{x+j}{j} \binom{p-1}{i} \binom{-p-1}{j} \\ & \quad \times \left( \frac{2p}{i+j+2} + \frac{i-j}{(i+j+1)(i+j+2)} \right) \\ & \equiv 0 \pmod{p}. \end{aligned} \quad (5.3)$$

Clearly,  $\binom{p-1}{i} \binom{-p-1}{j} \equiv (-1)^{i+j} \pmod{p}$  for any  $i, j = 0, 1, \dots, p-1$ . Thus

$$\begin{aligned} & \sum_{\substack{i, j \in \{0, \dots, p-1\} \\ i+j \neq p-1, p-2}} \binom{x}{i} \binom{x+i}{i} \binom{x}{j} \binom{x+j}{j} \binom{p-1}{i} \binom{-p-1}{j} \frac{i-j}{(i+j+1)(i+j+2)} \\ & \equiv \sum_{\substack{i, j \in \{0, \dots, p-1\} \\ i+j \neq p-1, p-2}} \binom{x}{i} \binom{x+i}{i} \binom{x}{j} \binom{x+j}{j} \frac{(-1)^{i+j}(i-j)}{(i+j+1)(i+j+2)} = 0 \pmod{p} \end{aligned}$$

by the symmetry of  $i$  and  $j$  in the last sum. If  $i+j = p-1$ , then

$$\binom{p-1}{i} \binom{-p-1}{j} = \binom{p-1}{j} \binom{-p-1}{j} = \prod_{0 < r \leq j} \left( 1 - \frac{p^2}{r^2} \right) \equiv 1 \pmod{p^2}.$$

So we also have

$$\begin{aligned} & \sum_{i+j=p-1} \binom{x}{i} \binom{x+i}{i} \binom{x}{j} \binom{x+j}{j} \binom{p-1}{i} \binom{-p-1}{j} \frac{i-j}{(i+j+1)(i+j+2)} \\ & \equiv \sum_{i+j=p-1} \binom{x}{i} \binom{x+i}{i} \binom{x}{j} \binom{x+j}{j} \frac{i-j}{p(p+1)} = 0 \pmod{p} \end{aligned}$$

by the symmetry of  $i$  and  $j$  in the last sum. If  $i+j = p-2$ , then by (2.9) we have

$$p \binom{p-1}{i} \binom{-p-1}{j} \frac{i-j}{(i+j+1)(i+j+2)} \equiv \frac{j-i-2p}{p-1} \equiv 2p + \frac{j-i}{p-1} \pmod{p^2}.$$

Thus

$$\begin{aligned} & p \sum_{i+j=p-2} \binom{x}{i} \binom{x+i}{i} \binom{x}{j} \binom{x+j}{j} \binom{p-1}{i} \binom{-p-1}{j} \frac{i-j}{(i+j+1)(i+j+2)} \\ & \equiv \sum_{i+j=p-2} \binom{x}{i} \binom{x+i}{i} \binom{x}{j} \binom{x+j}{j} \left( 2p + \frac{j-i}{p-1} \right) \\ & \equiv 2p \sum_{i+j=p-2} \binom{x}{i} \binom{x+i}{i} \binom{x}{j} \binom{x+j}{j} \pmod{p^2}. \end{aligned}$$

Note also that

$$\begin{aligned}
& \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \sum_{i+j=p-2} \binom{x}{i} \binom{x+i}{i} \binom{x}{j} \binom{x+j}{j} \binom{p-1}{i} \binom{-p-1}{j} \frac{p}{i+j+2} \\
& \equiv \sum_{i+j=p-2} \binom{x}{i} \binom{x+i}{i} \binom{x}{j} \binom{x+j}{j} \binom{p-1}{i} \binom{-p-1}{j} \\
& \equiv \sum_{i+j=p-2} \binom{x}{i} \binom{x+i}{i} \binom{x}{j} \binom{x+j}{j} (-1)^{i+j} \\
& = - \sum_{i+j=p-2} \binom{x}{i} \binom{x+i}{i} \binom{x}{j} \binom{x+j}{j} \pmod{p^2}.
\end{aligned}$$

Combining these we obtain (5.3). Thus (1.22) follows.

The proof of Theorem 1.4 is now complete.  $\square$

## 6. SOME FURTHER CONJECTURES

In this section we pose some further conjectures motivated by our results in Section 1.

Recall that a polynomial  $P(x)$  with real number coefficients is called *integer-valued* if  $P(x) \in \mathbb{Z}$  for all  $x \in \mathbb{Z}$ .

**Conjecture 6.1.** (i) *For any  $n \in \mathbb{Z}^+$ , the polynomial*

$$\frac{x(x+1)}{2n^2} \sum_{k=0}^{n-1} (2k+1) d_k(x)^2$$

*is integer-valued.*

(ii) *For any positive integers  $m$  and  $n$ , the polynomials*

$$\frac{1}{n} \sum_{k=0}^{n-1} (2k+1) d_k(x)^{2m} \quad \text{and} \quad \frac{1}{n} \sum_{k=0}^{n-1} (-1)^k (2k+1) d_k(x)^{2m}$$

*are integer-valued.*

**Conjecture 6.2.** *Let  $p > 3$  be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{4^k} d_k \left( -\frac{1}{6} \right)^2 \equiv \frac{p}{3} \left( \frac{p}{3} \right) \left( 4 \left( \frac{-2}{p} \right) - 1 \right) \pmod{p^2}.$$

*Also,*

$$\begin{aligned}
& \sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{4^k} d_k \left( -\frac{1}{3} \right)^2 \\
& \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 7 \pmod{24} \text{ \& } p = x^2 + 6y^2 \text{ } (x, y \in \mathbb{Z}), \\ 8x^2 - 2p \pmod{p^2} & \text{if } p \equiv 5, 11 \pmod{24} \text{ \& } p = 2x^2 + 3y^2 \text{ } (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } \left( \frac{-6}{p} \right) = -1. \end{cases}
\end{aligned}$$

When  $p > 5$  and  $(\frac{-6}{p}) = 1$ , we have

$$\sum_{k=0}^{p-1} (8k+3) \frac{\binom{2k}{k}}{4^k} d_k \left(-\frac{1}{3}\right)^2 \equiv 0 \pmod{p^2}.$$

*Remark 6.1.* Let  $p > 3$  be a prime. It is known (cf. [C]) that  $p = x^2 + 6y^2$  for some  $x, y \in \mathbb{Z}$  if  $p \equiv 1, 7 \pmod{24}$ , and  $p = 2x^2 + 3y^2$  for some  $x, y \in \mathbb{Z}$  if  $p \equiv 5, 11 \pmod{24}$ .

**Conjecture 6.3.** Let  $p > 3$  be a prime. Then

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{-1/2}{k} d_k \left(-\frac{1}{3}\right) d_k \left(-\frac{1}{6}\right) \\ & \equiv \begin{cases} (\frac{-1}{p})(4x^2 - 2p) \pmod{p^2} & \text{if } p \equiv 1, 7 \pmod{24} \text{ \& } p = x^2 + 6y^2 \ (x, y \in \mathbb{Z}), \\ (\frac{-1}{p})(2p - 8x^2) \pmod{p^2} & \text{if } p \equiv 5, 11 \pmod{24} \text{ \& } p = 2x^2 + 3y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } (\frac{-6}{p}) = -1. \end{cases} \end{aligned}$$

When  $(\frac{-6}{p}) = 1$ , we also have

$$\sum_{k=0}^{p-1} (64k+23) \binom{-1/2}{k} d_k \left(-\frac{1}{3}\right) d_k \left(-\frac{1}{6}\right) \equiv 4p \left(\frac{-1}{p}\right) \pmod{p^2}.$$

**Conjecture 6.4.** Let  $p > 3$  be a prime. Then

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{-1/3}{k} d_k \left(-\frac{1}{3}\right) \\ & \equiv \begin{cases} 2x - p/(2x) \pmod{p^2} & \text{if } p = x^2 + 3y^2 \ (x, y \in \mathbb{Z}) \text{ with } 3 \mid x-1, \\ 0 \pmod{p^3} & \text{if } p \equiv 2 \pmod{3}, \end{cases} \end{aligned}$$

and

$$\sum_{k=0}^{p-1} (4k+1) \binom{-1/3}{k} d_k \left(\frac{1}{3}\right) \equiv \frac{p}{6} \left(3 \left(\frac{p}{3}\right) - 1\right) \pmod{p^2}.$$

Moreover, for any integer  $n > 1$  we have

$$\frac{3^{3n-4}}{n} \sum_{k=0}^{n-1} (4k+1) \binom{-1/3}{k} d_k \left(-\frac{1}{3}\right) \in \mathbb{Z}.$$

**Conjecture 6.5.** *Let  $p > 3$  be a prime. Then*

$$\sum_{k=0}^{p-1} \binom{-2/3}{k} d_k \left(-\frac{2}{3}\right) \equiv \begin{cases} p \pmod{p^2} & \text{if } p \equiv 1 \pmod{3}, \\ -\frac{1}{3} \binom{(p+1)/2}{(p+1)/6} \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Also,

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{-1/6}{k} d_k \left(-\frac{1}{6}\right) \\ & \equiv \begin{cases} \left(\frac{-2}{p}\right) 2x \pmod{p} & \text{if } p = x^2 + 3y^2 \text{ } (x, y \in \mathbb{Z}) \text{ with } 3 \mid x-1, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases} \end{aligned}$$

**Conjecture 6.6.** *Let  $p$  be an odd prime. If  $p \equiv 3 \pmod{4}$ , then*

$$\sum_{k=0}^{p-1} \binom{-1/4}{k} d_k \left(-\frac{1}{4}\right) \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{4^k} d_k \left(-\frac{1}{4}\right)^2 \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{(-8)^k} d_k \left(-\frac{1}{4}\right)^2 \equiv 0 \pmod{p}.$$

If  $p \equiv 5, 7 \pmod{8}$ , then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{32^k} d_k \left(-\frac{1}{4}\right)^2 \equiv 0 \pmod{p}.$$

**Conjecture 6.7.** *Let  $p > 3$  be a prime. Then*

$$\sum_{n=0}^{p-1} \left( \sum_{k=0}^n \binom{n}{k} \frac{\binom{2k}{k}}{2^k} \right)^2 \equiv \sum_{n=0}^{p-1} \left( \sum_{k=0}^n \binom{n}{k} \frac{\binom{2k}{k}}{(-6)^k} \right)^2 \equiv \left( \frac{-1}{p} \right) \pmod{p^2}$$

and

$$\sum_{n=0}^{p-1} \left( \sum_{k=0}^n \binom{n}{k} \frac{\binom{2k}{k}}{2^k} \right) \left( \sum_{k=0}^n \binom{n}{k} \frac{\binom{2k}{k}}{(-6)^k} \right) \equiv \left( \frac{3}{p} \right) \pmod{p}.$$

Also,

$$\begin{aligned} & \sum_{n=0}^{p-1} n \left( \sum_{k=0}^n \binom{n}{k} \frac{\binom{2k}{k}}{2^k} \right)^2 \equiv \frac{3}{4} \left( \frac{3}{p} \right) p - \left( \frac{-1}{p} \right) \pmod{p^2}, \\ & \sum_{n=0}^{p-1} n \left( \sum_{k=0}^n \binom{n}{k} \frac{\binom{2k}{k}}{(-6)^k} \right)^2 \equiv -\frac{p}{4} \left( \frac{3}{p} \right) \pmod{p^2}, \\ & \sum_{n=0}^{p-1} \left( \sum_{k=0}^n \binom{n}{k} \frac{C_k}{2^k} \right)^2 \equiv 4 \left( \frac{-1}{p} \right) - 6 \left( \frac{3}{p} \right) + 3 \pmod{p}, \\ & \sum_{n=0}^{p-1} \left( \sum_{k=0}^n \binom{n}{k} \frac{C_k}{(-6)^k} \right)^2 \equiv 4 \left( \frac{-1}{p} \right) + 2 \left( \frac{3}{p} \right) - 5 \pmod{p}. \end{aligned}$$

*Remark 6.2.* By Ljunggren's identity (2.1) with  $x = -1/2$  and  $y = -2$ , for any  $n \in \mathbb{N}$  we have

$$\sum_{k=0}^n \binom{n}{k} \frac{\binom{2k}{k}}{2^k} = 3^n \sum_{k=0}^n \binom{n}{k} \frac{\binom{2k}{k}}{(-6)^k}.$$

We also observe that

$$\sum_{k=0}^{p-1} \frac{D_k(x, -2) D_k(-1-x, -2)}{3^k} \equiv 0 \pmod{p}$$

and

$$\sum_{k=0}^{p-1} \frac{D_k(x, -2) D_k(-1-x, -2)}{9^k} \equiv (-1)^{\langle x \rangle_p} \pmod{p}$$

for any prime  $p > 3$  and  $p$ -adic integer  $x$ .

**Conjecture 6.8.** *Let  $p > 3$  be a prime. Then*

$$\begin{aligned} & \sum_{n=0}^{p-1} \frac{\binom{2n}{n}}{4^n} \left( \sum_{k=0}^n \binom{n}{k} \frac{\binom{2k}{k}}{2^k} \right)^2 \\ & \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 7 \pmod{24} \text{ \& } p = x^2 + 6y^2 \ (x, y \in \mathbb{Z}), \\ 2p - 8x^2 \pmod{p^2} & \text{if } p \equiv 5, 11 \pmod{24} \text{ \& } p = 2x^2 + 3y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } \left(\frac{-6}{p}\right) = -1. \end{cases} \end{aligned}$$

When  $\left(\frac{-6}{p}\right) = 1$ , we have

$$\sum_{n=0}^{p-1} (48n + 25) \frac{\binom{2n}{n}}{4^n} \left( \sum_{k=0}^n \binom{n}{k} \frac{\binom{2k}{k}}{2^k} \right)^2 \equiv 8p \left( \frac{p}{3} \right) \pmod{p^2}.$$

**Conjecture 6.9.** *Let  $p > 3$  be a prime. Then*

$$\begin{aligned} & \sum_{n=0}^{p-1} \frac{\binom{2n}{n}^2}{(-2)^n} \sum_{k=0}^n \binom{n}{k} \frac{\binom{2k}{k}}{2^k} \\ & \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{3} \text{ \& } p = x^2 + 3y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases} \end{aligned}$$

Also,

$$\sum_{n=0}^{p-1} (5n + 2) \frac{\binom{2n}{n}^2}{(-2)^n} \sum_{k=0}^n \binom{n}{k} \frac{\binom{2k}{k}}{2^k} \equiv \frac{2p}{3} \left( 1 + 2 \left( \frac{-1}{p} \right) \right) \pmod{p^2},$$

$$\sum_{n=0}^{p-1} \frac{\binom{2n}{n}^2}{64^n} \sum_{k=0}^n \binom{n}{k} \frac{\binom{2k}{k}}{2^k} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}, \\ 0 \pmod{p} & \text{if } p \equiv 5 \pmod{12}, \end{cases}$$

and

$$\sum_{n=0}^{p-1} \frac{\binom{4n}{2n} \binom{2n}{n}}{256^n} \sum_{k=0}^n \binom{n}{k} \frac{\binom{2k}{k}}{2^k} \equiv 0 \pmod{p^2} \quad \text{if } p \equiv 5, 7 \pmod{8}.$$

*Remark 6.3.* It is well known that any prime  $p \equiv 1 \pmod{3}$  can be written as  $x^2 + 3y^2$  with  $x, y \in \mathbb{Z}$ .

**Conjecture 6.10.** For any prime  $p > 3$  and  $p$ -adic integer  $x \neq -1/2$ , we have the congruence

$$\sum_{k=0}^{p-1} s_k(x)^2 \equiv (-1)^{\langle x \rangle_p} \frac{p + 2(x - \langle x \rangle_p)}{2x + 1} \pmod{p^3}.$$

*Remark 6.4.* This is a further refinement of the congruences (1.20) and (1.21) in Theorem 1.4.

Let  $B_0, B_1, B_2, \dots$  be the Bernoulli numbers and let  $B_n(x) = \sum_{k=0}^n \binom{n}{k} B_k x^{n-k}$  be the Bernoulli polynomial of degree  $n$ .

**Conjecture 6.11.** Let  $p$  be an odd prime. Then

$$\begin{aligned} \sum_{k=0}^{p-1} s_k \left( -\frac{1}{2} \right)^2 &\equiv \left( \frac{-1}{p} \right) (1 - 7p^3 B_{p-3}) \pmod{p^4}, \\ \sum_{k=0}^{p-1} (2k+1) s_k \left( -\frac{1}{2} \right)^2 &\equiv \frac{3}{4} \left( \frac{-1}{p} \right) p^2 \pmod{p^4}, \\ \sum_{k=0}^{p-1} s_k \left( -\frac{1}{4} \right)^2 &\equiv \left( \frac{2}{p} \right) p - 26 \left( \frac{-2}{p} \right) p^3 E_{p-3} \pmod{p^4}, \\ \sum_{k=0}^{p-1} (2k+1) s_k \left( -\frac{1}{4} \right)^2 &\equiv \frac{13}{16} \left( \frac{-2}{p} \right) p^2 \pmod{p^4}. \end{aligned}$$

When  $p > 3$ , we also have

$$\begin{aligned} \sum_{k=0}^{p-1} s_k \left( -\frac{1}{3} \right)^2 &\equiv p - \frac{14}{3} \left( \frac{p}{3} \right) p^3 B_{p-2} \left( \frac{1}{3} \right) \pmod{p^4}, \\ \sum_{k=0}^{p-1} (2k+1) s_k \left( -\frac{1}{3} \right)^2 &\equiv \frac{7}{9} \left( \frac{p}{3} \right) p^2 \pmod{p^4}, \\ \sum_{k=0}^{p-1} s_k \left( -\frac{1}{6} \right)^2 &\equiv \left( \frac{3}{p} \right) p - \frac{155}{12} \left( \frac{-1}{p} \right) p^3 B_{p-2} \left( \frac{1}{3} \right), \\ \sum_{k=0}^{p-1} (2k+1) s_k \left( -\frac{1}{6} \right)^2 &\equiv \frac{31}{36} \left( \frac{-1}{p} \right) p^2 \pmod{p^4}. \end{aligned}$$



**Conjecture 6.12.** *For any  $m, n \in \mathbb{Z}^+$ , the polynomials*

$$\frac{1}{n} \sum_{k=0}^{n-1} (2k+1) s_k(x)^{2m} \quad \text{and} \quad \frac{1}{n} \sum_{k=0}^{n-1} (2k+1) (-1)^k s_k(x)^{2m}$$

*are integer-valued. Also, for any positive integer  $n$ , all the polynomials*

$$\begin{aligned} & \frac{1}{n^2} \sum_{k=0}^{n-1} (2k+1) s_k(x)^2, \\ & \frac{1}{n^2} \sum_{k=0}^{n-1} (2k+1) s_k(x) \sum_{j=0}^{n-1} \binom{x}{j} \binom{-1-x}{j} \binom{k+j}{j}, \\ & \frac{1}{n^2} \sum_{k=0}^{n-1} (2k+1) \left( \sum_{j=0}^{n-1} \binom{x}{j} \binom{-1-x}{j} \binom{k+j}{j} \right)^2 \\ & \frac{2}{n(n+1)} \sum_{k=0}^{n-1} (k+1) \left( \sum_{j=0}^k \binom{k}{j} \binom{x}{j} \binom{-1-x}{j} \frac{(-1)^j}{j+1} \right)^2 \\ & \frac{2}{n(n+1)} \sum_{k=0}^{n-1} (k+1) \left( \sum_{j=0}^k \binom{x}{j} \binom{-1-x}{j} \binom{k+j+1}{j+1} \right)^2 \end{aligned}$$

*are integer-valued.*

**Conjecture 6.13.** *Let  $p$  be an odd prime. Then*

$$\begin{aligned} \sum_{n=0}^{p-1} \left( \sum_{k=0}^n \frac{\binom{2k}{k} C_k}{(-16)^k} \right)^2 &\equiv 4 \left( \frac{-1}{p} \right) + 3p^2 \left( 3 - 4 \left( \frac{-1}{p} \right) \right) \pmod{p^3}, \\ \sum_{n=0}^{p-1} (n+1) \left( \sum_{k=0}^n \binom{n}{k} \frac{\binom{2k}{k} C_k}{(-16)^k} \right)^2 &\equiv \frac{3}{2} \left( \frac{-1}{p} \right) p^2 + 3p^3 \left( 3 - 4 \left( \frac{-1}{p} \right) \right) \pmod{p^4}. \end{aligned}$$

*If  $p > 3$ , then*

$$\begin{aligned} \sum_{n=0}^{p-1} \left( \sum_{k=0}^n \frac{\binom{3k}{k} C_k}{(-27)^k} \right)^2 &\equiv \frac{9}{2} p + \frac{7}{4} p^2 \left( 7 - 9 \left( \frac{p}{3} \right) \right) \pmod{p^3}, \\ \sum_{n=0}^{p-1} (n+1) \left( \sum_{k=0}^n \binom{n}{k} \frac{\binom{3k}{k} C_k}{(-27)^k} \right)^2 &\equiv \frac{7}{4} \left( \frac{p}{3} \right) p^2 + \frac{7}{4} p^3 \left( 7 - 9 \left( \frac{p}{3} \right) \right) \pmod{p^4}, \\ \sum_{n=0}^{p-1} \left( \sum_{k=0}^n \frac{\binom{4k}{2k} C_k}{(-64)^k} \right)^2 &\equiv \frac{16}{3} \left( \frac{2}{p} \right) p + \frac{13}{9} p^2 \left( 13 - 16 \left( \frac{-2}{p} \right) \right) \pmod{p^3}, \\ \sum_{n=0}^{p-1} (n+1) \left( \sum_{k=0}^n \binom{n}{k} \frac{\binom{4k}{2k} C_k}{(-64)^k} \right)^2 &\equiv \frac{13}{6} \left( \frac{-2}{p} \right) p^2 + \frac{13}{9} p^3 \left( 13 - 16 \left( \frac{-2}{p} \right) \right) \pmod{p^4}. \end{aligned}$$

**Conjecture 6.14.** For  $n \in \mathbb{N}$  define

$$t_n(x) := S_n(x, -2) = \sum_{k=0}^n \binom{n}{k} \binom{x}{k} \binom{x+k}{k} 2^k.$$

(i) For any odd prime  $p$  and  $p$ -adic integer  $x$ , we have

$$\sum_{k=0}^{p-1} t_k(x)^2 \equiv \begin{cases} \left(\frac{-1}{p}\right) \pmod{p^2} & \text{if } 2x \equiv -1 \pmod{p}, \\ (-1)^{\langle x \rangle_p} (p + 2x - 2\langle x \rangle_p) / (2x + 1) \pmod{p^2} & \text{otherwise.} \end{cases}$$

(ii) For any  $n \in \mathbb{Z}^+$  and  $x \in \mathbb{Z}$ , the number

$$\frac{1}{n} \sum_{k=0}^{n-1} (8k + 5) t_k(x)^2$$

is always an odd integer.

(iii) Let  $p$  be an odd prime. Then

$$\begin{aligned} \sum_{k=0}^{p-1} (8k + 5) t_k \left(-\frac{1}{2}\right)^2 &\equiv 2p \pmod{p^2}, \\ \sum_{k=0}^{p-1} (32k + 21) t_k \left(-\frac{1}{4}\right)^2 &\equiv 8p \pmod{p^2}. \end{aligned}$$

If  $p > 3$ , then

$$\begin{aligned} \sum_{k=0}^{p-1} (18k + 7) t_k \left(-\frac{1}{3}\right)^2 &\equiv 0 \pmod{p^2}, \\ \sum_{k=0}^{p-1} (72k + 49) t_k \left(-\frac{1}{6}\right)^2 &\equiv 18p \pmod{p^2}. \end{aligned}$$

*Remark 6.5.* Part (i) of Conjecture 6.14 with  $x = -1/2, -1/4, -1/3, -1/6$  yields the following congruences for odd primes  $p$ :

$$\begin{aligned} \sum_{n=0}^{p-1} \left( \sum_{k=0}^n \binom{n}{k} \frac{\binom{2k}{k}^2}{(-8)^k} \right)^2 &\equiv \left(\frac{-1}{p}\right) \pmod{p^2}, \\ \sum_{n=0}^{p-1} \left( \sum_{k=0}^n \binom{n}{k} \frac{\binom{2k}{k} \binom{4k}{2k}}{(-32)^k} \right)^2 &\equiv \left(\frac{2}{p}\right) p \pmod{p^2}, \\ \sum_{n=0}^{p-1} \left( \sum_{k=0}^n \binom{n}{k} \frac{2^k \binom{2k}{k} \binom{3k}{k}}{(-27)^k} \right)^2 &\equiv p \pmod{p^2} \quad (p \neq 3), \\ \sum_{n=0}^{p-1} \left( \sum_{k=0}^n \binom{n}{k} \frac{\binom{3k}{k} \binom{6k}{3k}}{(-216)^k} \right)^2 &\equiv \left(\frac{3}{p}\right) p \pmod{p^2} \quad (p \neq 3). \end{aligned}$$

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